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Problem of the Week #1 — Solution

A Rolle's theorem argument via an antiderivative construction

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ABSTRACT

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Every continuous function $f : [0, 1] \to \mathbb{R}$ satisfying $\int_0^1 f(x)\,\mathrm{d}x = 1$ admits a point $c \in (0, 1)$ at which $f(c) = 2c$. The proof constructs the auxiliary function $g(x) = \int_0^x f(t)\,\mathrm{d}t - x^2$ and applies *Rolle's theorem* after establishing that g is continuously differentiable on $(0, 1)$. The integral hypothesis is precisely the condition that forces $g(1) = 0$, thereby supplying the equal-endpoint boundary condition on which the entire argument rests. Documented errors include misapplication of the intermediate value theorem and failure to verify all three hypotheses of Rolle's theorem in full. The result generalises: if $\int_0^1 f = k$, then $f(c) = 2kc$ for some $c \in (0, 1)$.

1. PROBLEM STATEMENT

PROBLEM ML-POW-2026-001 · UNDERGRADUATE · WEEK OF 2026-04-07

Let $f : [0, 1] \to \mathbb{R}$ be a continuous function satisfying

$$\int_0^1 f(x)\,\mathrm{d}x = 1. \tag{1}$$

Prove that there exists a point $c \in (0, 1)$ such that

$$f(c) = 2c. \tag{2}$$

Background and motivation

The conclusion (2) asserts that the graph of f meets the line $y = 2x$ at some interior point of $(0, 1)$, a result belonging to a classical family in real analysis where global integral constraints force the existence of points satisfying local pointwise identities. The condition (1) imposes a precise global balance on f over $[0, 1]$: it is exactly the value that equalises the auxiliary antiderivative $g(x) = \int_0^x f(t)\,\mathrm{d}t - x^2$ at the two endpoints, and it is this equal-endpoint property that activates Rolle's theorem. The technique of introducing an antiderivative to convert an integral constraint into a boundary condition, and then differentiating, recurs throughout analysis: it underpins the mean value theorem for integrals, comparison theorems for second-order ODEs, and classical uniqueness arguments in boundary value problems. The problem is a first illustration of the principle that hypotheses stated in terms of $\int_0^1 f$ should be converted into constraints on $\int_0^x f$ via the Fundamental Theorem of Calculus, which serves as the bridge between the integral and the differential regimes.

2. SOLUTION

SOLUTION

Step 1: Construction of the auxiliary function. Define $g : [0,1] \to \mathbb{R}$ by

$$g(x) := \int_0^x f(t)\, \mathrm{d}t - x^2. \tag{3}$$

Since f is continuous on $[0,1]$, the *Fundamental Theorem of Calculus* [1] guarantees that the map $x \mapsto \int_0^x f(t)\, \mathrm{d}t$ is continuous on $[0,1]$. The function $x \mapsto x^2$ is a polynomial, hence continuous on $[0,1]$ and differentiable on $(0,1)$. Therefore g is continuous on $[0,1]$ and differentiable on $(0,1)$.

Step 2: Verification of Rolle's theorem hypotheses. The values of g at the endpoints are computed as follows:

$$g(0) = \int_0^0 f(t)\, \mathrm{d}t - 0^2 = 0, \tag{4}$$

$$g(1) = \int_0^1 f(t)\, \mathrm{d}t - 1^2. \tag{5}$$

The hypothesis (1) gives $\int_0^1 f(t)\, \mathrm{d}t = 1$; substituting into (5),

$$g(1) = 1 - 1 = 0.$$

Hence $g(0) = g(1) = 0$.

The three hypotheses of the following theorem are now verified for g on $[0,1]$.

Theorem 2.1 (*Rolle* [1]). *Let $h : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), and suppose $h(a) = h(b)$. Then there exists a point $\xi \in (a,b)$ such that $h'(\xi) = 0$.*

Applying this theorem to g on $[0,1]$: (i) g is continuous on $[0,1]$ (Step 1); (ii) g is differentiable on $(0,1)$ (Step 1); (iii) $g(0) = g(1) = 0$ (computed above, using (1) at (5)). Therefore there exists a point $c \in (0,1)$ such that

$$g'(c) = 0. \tag{6}$$

Step 3: Conclusion via the derivative. Differentiating (3) with respect to x, the *Fundamental Theorem of Calculus, Part 1* [1] gives

$$g'(x) = f(x) - 2x, \qquad x \in (0,1), \tag{7}$$

since $\frac{\mathrm{d}}{\mathrm{d}x} \int_0^x f(t)\, \mathrm{d}t = f(x)$ (as f is continuous) and $\frac{\mathrm{d}}{\mathrm{d}x}(x^2) = 2x$. Evaluating (7) at the point c furnished by (6),

$$0 = g'(c) = f(c) - 2c. \tag{8}$$

Rearranging (8) yields $f(c) = 2c$, which is precisely the conclusion (2). ■

■

3. REMARKS AND INSIGHTS

Remark (Role of the integral condition). The value $\int_0^1 f = 1$ is not an arbitrary normalisation; it is the unique constant that causes the auxiliary function $g(x) = \int_0^x f(t)\ \mathrm{d}t - x^2$ to satisfy $g(1) = 0$, thereby meeting the equal-endpoint requirement of Rolle's theorem. The argument generalises cleanly: suppose $f : [0,1] \to \mathbb{R}$ is continuous with $\int_0^1 f(x)\ \mathrm{d}x = k$ for some constant $k \in \mathbb{R}$. Define

$$g_k(x) := \int_0^x f(t)\ \mathrm{d}t - kx^2.$$

Then $g_k(0) = 0$ and $g_k(1) = k - k \cdot 1^2 = 0$, so the equal-endpoint condition holds for every k. Since g_k is continuous on $[0,1]$ and differentiable on $(0,1)$ by the same FTC argument, Rolle's theorem supplies $c \in (0,1)$ with $g_k'(c) = 0$. Computing $g_k'(x) = f(x) - 2kx$, the conclusion is: *if* $\int_0^1 f = k$, *then* $f(c) = 2kc$ *for some* $c \in (0,1)$. The case $k = 0$ asserts $f(c) = 0$, i.e. a continuous function with mean zero has an interior zero. The original problem is the instance $k = 1$.

Remark (Why Rolle's theorem, not the mean value theorem). *Rolle's theorem* is the special case of the *mean value theorem* (MVT) in which the equal-endpoint condition $h(a) = h(b)$ reduces the average rate of change to zero: if $g(0) = g(1)$, then the MVT gives $g'(\xi) = (g(1) - g(0))/(1 - 0) = 0$, recovering Rolle's conclusion. In this problem both formulations are therefore logically equivalent. However, Rolle's theorem is the cleaner tool for two reasons. First, the equal-endpoint condition $g(0) = g(1) = 0$ is precisely what the hypothesis $\int_0^1 f = 1$ provides: the proof presents itself as a verification of Rolle's three hypotheses, with each hypothesis corresponding to a transparent piece of the argument. Second, invoking the full MVT would require computing $(g(1) - g(0))/(1 - 0)$, which is simply $0/1 = 0$ by the equal-endpoint condition, adding a trivial calculation that obscures the structure. The principle to retain: reach for Rolle's theorem whenever the natural auxiliary function satisfies $g(a) = g(b)$; reserve the full MVT for situations where $g(a) \neq g(b)$ and the average rate of change carries information.

Remark (Geometric interpretation). The identity $f(c) = 2c$ asserts that the graph of f passes through the point $(c, 2c)$, which lies on the line $\ell : y = 2x$. The problem therefore asks for a point of intersection between the graph of f and the line ℓ. The integral condition (1) guarantees this intersection exists via the following observation: if $f(x) > 2x$ for all $x \in (0,1)$, then by continuity $f(x) \geq 2x$ on all of $[0,1]$, so

$$\int_0^1 f(x)\ \mathrm{d}x \ > \ \int_0^1 2x\ \mathrm{d}x = 1,$$

contradicting (1); symmetrically, $f < 2x$ throughout leads to $\int_0^1 f < 1$. Setting $h(x) = f(x) - 2x$, the function h is continuous with $\int_0^1 h\ \mathrm{d}x = 0$, and neither $h > 0$ nor $h < 0$ can hold throughout $(0,1)$ without violating this integral condition. Therefore h must change sign (or be identically zero) on $(0,1)$, and in either case the intermediate value theorem produces $c \in (0,1)$ with $h(c) = 0$. This geometric reasoning constitutes a second, independent proof of the result; the Rolle's theorem argument in Section 2 is preferred for its directness and its explicit use of the derivative.

Remark (Uniqueness of the point c***).*** The point $c \in (0,1)$ satisfying $f(c) = 2c$ need not be unique. The simplest example is $f(x) = 2x$: since $\int_0^1 2x\ \mathrm{d}x = \left[x^2\right]_0^1 = 1$, the integral condition (1) is satisfied, and $f(c) = 2c$ holds for *every* $c \in [0,1]$, giving infinitely many solutions. For a less degenerate example with multiple interior solutions, consider

$$f(x) := 2x + \sin(4\pi x).$$

The integral condition holds since $\int_0^1 \sin(4\pi x)\ \mathrm{d}x = \left[-\frac{\cos(4\pi x)}{4\pi}\right]_0^1 = 0$, so $\int_0^1 f = \int_0^1 2x\ \mathrm{d}x = 1$. The equation $f(c) = 2c$ reduces to $\sin(4\pi c) = 0$, which yields three interior solutions $c = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$. More generally, $f(x) = 2x + \sin(2m\pi x)$ for any positive integer m satisfies $\int_0^1 f = 1$ (since $\int_0^1 \sin(2m\pi x)\ \mathrm{d}x = 0$) and produces $2m - 1$ interior solutions $c = k/(2m)$, $k = 1, \ldots, 2m - 1$. Thus the number of solutions can be made arbitrarily large while the integral condition remains satisfied.

4. COMMON ERRORS

Error 1: Applying the intermediate value theorem directly

A natural first attempt defines $h(x) := f(x) - 2x$ and seeks to apply the *intermediate value theorem* by finding points $x_0, x_1 \in [0,1]$ at which $h(x_0) \leq 0 \leq h(x_1)$. This strategy requires explicit sign information about h at specific points, typically $h(0) = f(0)$ and $h(1) = f(1) - 2$. The integral condition (1) does not supply this: it constrains the average of f over $[0,1]$ but places no restriction on the pointwise values $f(0)$ or $f(1)$. For instance, $f(x) = 3x^2$ satisfies $\int_0^1 3x^2 \, \mathrm{d}x = 1$ and has $h(0) = 0$, $h(1) = 1 > 0$; the function $f(x) = \frac{3}{2}\sqrt{x}$ satisfies the integral condition with $h(0) = 0$ as well. In neither case does h change sign at the endpoints in the manner required for a direct IVT argument. The correct resolution is to abandon the direct approach on f and instead introduce the antiderivative $g(x) = \int_0^x f(t) \, \mathrm{d}t - x^2$, for which the integral condition (1) translates into the boundary condition $g(1) = 0$ required by Rolle's theorem.

Error 2: Invoking Rolle's theorem without verifying all three hypotheses

Several solutions stated "by Rolle's theorem, there exists $c \in (0,1)$ with $g'(c) = 0$" immediately after defining g, without verifying the three prerequisite hypotheses. All three must be checked explicitly.

(i) *Continuity of g on $[0,1]$:* Since f is continuous, the Fundamental Theorem of Calculus guarantees that $x \mapsto \int_0^x f(t) \, \mathrm{d}t$ is continuous; the subtraction of x^2 preserves continuity. This step uses the continuity hypothesis on f; a merely integrable f would not be sufficient to conclude differentiability in step (ii).

(ii) *Differentiability of g on $(0,1)$:* FTC Part 1 applies because f is continuous, yielding $g'(x) = f(x) - 2x$ at every $x \in (0,1)$.

(iii) *Equal endpoint values, $g(0) = g(1)$:* This is the step that consumes the hypothesis (1). Writing $g(1) = \int_0^1 f(t) \, \mathrm{d}t - 1 = 1 - 1 = 0$ requires the given condition; without it, $g(1)$ is not zero and Rolle's theorem cannot be applied.

Omitting any of these verifications constitutes an incomplete proof, regardless of whether the conclusion happens to be correct.

Error 3: Differentiating g incorrectly

Two distinct differentiation errors were observed.

Error 3a: writing $g'(x) = f'(x) - 2x$. This conflates $\frac{\mathrm{d}}{\mathrm{d}x} \int_0^x f(t) \, \mathrm{d}t$ with $\int_0^x f'(t) \, \mathrm{d}t$. The Fundamental Theorem of Calculus, Part 1 states that if f is continuous then

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_0^x f(t) \, \mathrm{d}t = f(x),$$

not $f'(x)$. The variable x appears only in the upper limit of integration, not in the integrand; differentiation extracts the integrand evaluated at the upper limit.

Error 3b: writing $g'(x) = f(x) - 2$. This results from differentiating x^2 as though it were $2x$, confusing $\frac{\mathrm{d}}{\mathrm{d}x}(x^2) = 2x$ with $\frac{\mathrm{d}}{\mathrm{d}x}(2x) = 2$. The correct derivative is $\frac{\mathrm{d}}{\mathrm{d}x}(x^2) = 2x$, yielding $g'(x) = f(x) - 2x$. Setting $g'(c) = f(c) - 2 = 0$ would give $f(c) = 2$, a constant, which is not the required conclusion (2) and does not follow from the given hypotheses.

5. CORRECT SUBMISSIONS

REFERENCES

[1] Walter Rudin. *Principles of Mathematical Analysis.* McGraw-Hill, New York, 3rd edition, 1976.

